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Finite groups whose subgroup graph contains a vertex of large degree

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Abstract. Burness and Scott (J Aust Math Soc 87:329-357, 2009) classified finite groups G such that the number of prime order subgroups of G is greater than $|G|/2-1$. In this note, we study finite groups G whose subgroup graph contains a vertex of degree greater than $|G|/2-1$. The classification given for finite solvable groups extends the work of Burness and Scott.

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1. Introduction. Recently, mathematicians constructed many graphs which are assigned to groups by different methods (see e.g. [4]). One of them is the *subgroup graph* $L(G)^*$ of a group G, that is, the graph whose vertices are the subgroups of G and two vertices H_1 and H_2 are connected by an edge if and only if $H_1 \leq H_2$ and there is no subgroup $K \leq G$ such that $H_1 < K < H_2$. Finite groups with planar subgroup graph have been determined by Starr and Turner [12], Bohanon and Reid [1], and Schmidt [10]. Also, the genus of $L(G)^*$ has been studied by Lucchini [8]. A remarkable subgraph of $L(G)^*$ is the *cyclic subgroup graph* of G, which has been investigated in [13]. Note that the (cyclic) subgroup graph of a group G is in fact the Hasse diagram of the poset of (cyclic) subgroups of G, viewed as a simple, undirected graph.

The starting point for our discussion is given by the paper [3], which classifies the nontrivial finite groups G with the property $\delta(G) > |G|/2-1$, where $\delta(G)$ is the number of prime order subgroups of G. Recall that a *generalized dihedral group* is a group of the form $D(A) = A\langle\tau\rangle = A.2$, where A is abelian and τ acts by inversion. We mention that E, C_n, D_8 denote an elementary

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abelian 2-group, a cyclic group of order n, and the dihedral group of order 8, respectively.

Theorem A. *For a nontrivial finite group G, we have $\delta(G) > |G|/2 - 1$ if and only if one of the following holds:*

(I) $G = D(A)$, *where A is abelian;*
(II) $G = D_8 \times D_8 \times E$, *where E is an elementary abelian 2-group;*
(III) $G = H(r) \times E$, *where $H(r) \cong (D_8 \times \cdots \times D_8)/C_2^{r-1}$ is a central product of $r \geq 1$ copies of D_8 so that*

$$H(r) = \langle x_1, y_1, \ldots, x_r, y_r, z \mid x_i^2 = y_i^2 = z^2 = 1, \text{ all pairs of generators commute except } [x_i, y_i] = z\rangle;$$

(IV) $G = S(r) \times E$, *where $S(r)$ is the split extension of an elementary abelian group of order 2^{2r} $(r \geq 1)$ by a cyclic group $C_2 = \langle z\rangle$ so that*

$$S(r) = \langle x_1, y_1, \ldots, x_r, y_r, z \mid x_i^2 = y_i^2 = z^2 = 1, \text{ all pairs of generators commute except } [z, x_i] = x_i y_i\rangle;$$

(V) $G = T(r)$ *is the split extension of an elementary abelian group A of order 2^{2r} $(r \geq 1)$ by a cyclic group $C_3 = \langle z\rangle$ so that*

$$T(r) = \langle x_1, y_1, \ldots, x_r, y_r, z \mid x_i^2 = y_i^2 = z^3 = 1, \text{ all pairs of generators commute except } [z, x_i] = x_i y_i \text{ and } [z, y_i] = x_i\rangle;$$

(VI) G *is a group of exponent* 3;
(VII) $G = S_3 \times D_8 \times E$;
(VIII) $G = S_3 \times S_3$;
(IX) $G = S_4$;
(X) $G = A_5$.

In [14], C.T.C. Wall classifies the finite groups G with the property $i_2(G) > |G|/2 - 1$, where $i_2(G)$ is the number of involutions in G. Since $\delta(G) \geq i_2(G)$, Theorem A generalizes Wall's result. More precisely, the groups obtained by Wall are those of types (I)-(IV) in Theorem A.

We remark that $\delta(G)$ is the degree of the trivial subgroup in the subgroup graph of G. Thus a natural extension of Theorem A is to classify the finite groups G such that $d_{L(G)^*}(H) > |G|/2 - 1$ for some subgroup H of G.

Our main result is as follows.

Theorem 1.1. *Let G be a nontrivial finite solvable group whose subgroup graph contains a vertex of degree strictly greater than $|G|/2 - 1$. Then G belongs to at least one of the following families of groups:*

1) *The groups of order at most* 11 *except C_n for $n = 5, \ldots, 11$.*
2) *The groups (I)-(IX) in Theorem A.*
3) *Elementary abelian* 2*-groups.*
4) $C_2^{s-1} \times C_4$, *where $s \geq 1$.*
5) *Generalized extraspecial* 2*-groups.*
6) $C_p^n \rtimes C_2$, *where p is an odd prime and n is a positive integer.*
7) D_{12}.

Note that, among these groups, there are two families not contained in Theorem A: $C_2^{s-1} \times C_4$ with $s \geq 1$ and generalized extraspecial 2-groups which are not of type (III) in reference to the classification of Theorem A. Recall that a finite 2-group is called *generalized extraspecial* if $G' = \Phi(G)$ has order 2 and $G' \subseteq Z(G)$. Here G', $\Phi(G)$, $Z(G)$ denote the derived subgroup, the Frattini subgroup, and the centre of G, respectively. The structure of these groups is well-known (see e.g. Theorem 2.3 of [2] and Lemma 3.2 of [11]).

We will prove that for a solvable group G, if a vertex H of the subgroup graph $L(G)^*$ has "large degree" (i.e. $d_{L(G)^*}(H) > |G|/2 - 1$), then this vertex lies at the very top or at the very bottom of the Hasse diagram of the subgroup lattice of G. For the proof of Theorem 1.1, we need the following theorem which collects some results on the number of maximal subgroups of a finite solvable group.

Theorem B. *Let G be a finite solvable group,* $\mathrm{Max}(G)$ *be the set of maximal subgroups of G, and $\pi(G)$ be the set of primes dividing the order of G. For every $p \in \pi(G)$, we denote by $\mathrm{Max}_p(G)$ the set of maximal subgroups of G whose index is a power of p and by $\mathbf{O}^p(G)$ the smallest normal subgroup of G whose index is a power of p. Then the following statements hold:*

(a) (G.E. Wall [15]) $|\mathrm{Max}(G)| \leq |G| - 1$.

(b) (R.J. Cook, J. Wiegold, and A.G. Williamson [5]) *If $p = \min \pi(G)$, then*

$$|\mathrm{Max}(G)| \leq \frac{|G| - 1}{p - 1}.$$

Moreover, we have equality if and only if G is elementary abelian.

(c) (M. Herzog and O. Manz [6]) *If $p = \min \pi(G)$ and $q = \max \pi(G)$, then*

$$|\mathrm{Max}(G)| \leq \frac{q|G/\Phi(G)| - p}{p(q - 1)}.$$

(d) (B. Newton [9]) *If $|G| = p^k m$, where $p \nmid m$, and $[G : \mathbf{O}^p(G)] = p^r$, then*

$$|\mathrm{Max}_p(G)| \leq \frac{p^r - 1}{p - 1} + \frac{p^{k-r+1} - p}{p - 1}.$$

Moreover, $|\mathrm{Max}_p(G)| \leq \frac{p^{k+1} - p}{p - 1}$, and if $\mathbf{O}^p(G) < G$, then

$$|\mathrm{Max}_p(G)| \leq \frac{p^k - 1}{p - 1}.$$

(e) (B. Newton [9]) *If $|G| = p_1^{n_1} \cdots p_u^{n_u}$ is the decomposition of $|G|$ as a product of distinct primes and $p_1^{n_1} = \min\{p_i^{n_i} : 1 \leq i \leq u\}$, then*

$$|\mathrm{Max}(G)| \leq \frac{p_1^{n_1} - 1}{p_1 - 1} + \sum_{i=2}^{u} \frac{p_i^{n_i+1} - p_i}{p_i - 1}.$$

The next corollaries follow immediately from the proof of Theorem 1.1 and Corollaries 1.2 and 1.5 of [3].

Corollary 1.1. *Let G be a finite solvable group. Then the subgroup graph of G contains a vertex of degree at least $3|G|/4$ if and only if G is an elementary abelian 2-group.*

Corollary 1.2. *Let G be a finite solvable group. Then the subgroup graph of G contains a vertex of degree $|G|/2$ if and only if one of the following holds:*

(1) $G = S_3 \times D_8 \times E$ *with* $\exp(E) \leq 2$.
(2) G *is an elementary abelian* 2*-group.*
(3) $G = C_2^{s-1} \times C_4$*, where* $s \geq 1$.
(4) G *is a generalized extraspecial* 2*-group.*

Inspired by these results, we formulate the following natural open problems.

Open problems.

1. Given a constant $c \in (0, 1)$, classify finite groups G containing a subgroup H with $d_{L(G)^*}(H) \geq c|G|$.
2. Classify *arbitrary* finite groups G whose subgroup graph contains a vertex of degree greater than $|G|/2 - 1$.

Most of our notation is standard and will usually not be repeated here. For basic notions and results on groups, we refer the reader to [7].

2. Proof of the main result. We start by proving two auxiliary results. The first one gives an upper bound for the degree of a vertex H in the subgroup graph of a finite solvable group G. It depends on the orders of H and G.

Lemma 2.1. *Let G be a finite solvable group of order n and H be a subgroup of order d of G. Then*

$$d_{L(G)^*}(H) \leq d + \frac{n}{d} - 2. \tag{1}$$

Moreover, we have equality if and only if H is normal in G and both H and G/H are elementary abelian 2*-groups.*

Proof. If $k = |\mathrm{Max}(H)|$, then, by Theorem B (a), we have

$$k \leq d - 1. \tag{2}$$

Let $N_1, \ldots, N_r$ be the atoms of the lattice interval between H and G. Then $|N_i| \geq 2|H|$ and so $|N_i \setminus H| \geq |H|$ for all $i = 1, \ldots, r$. We get

$$\begin{aligned} n - d = |G \setminus H| &\geq \left| \left(\bigcup_{i=1}^{r} N_i \right) \setminus H \right| = \left| \bigcup_{i=1}^{r} (N_i \setminus H) \right| = \sum_{i=1}^{r} |N_i \setminus H| \\ &\geq r|H| = rd, \end{aligned}$$

thus

$$r \leq \frac{n}{d} - 1. \tag{3}$$

It is now clear that inequalities (2) and (3) lead to

$$d_{L(G)^*}(H) = k + r \leq d + \frac{n}{d} - 2,$$

as desired.

The second part of the proof uses Theorem B (b). If H is normal in G and both H and G/H are elementary abelian 2-groups, then $k = d - 1$ and

$r = \frac{n}{d} - 1$, implying that $d_{L(G)^*}(H) = d + \frac{n}{d} - 2$. Conversely, assume that $d_{L(G)^*}(H) = d + \frac{n}{d} - 2$. Then

$$k = d - 1 \text{ and } r = \frac{n}{d} - 1. \tag{4}$$

The first equality in (4) shows that H is an elementary abelian 2-group. From the second one, we infer that $|N_i| = 2|H|$ for all $i = 1, \ldots, r$, and $G = \bigcup_{i=1}^{r} N_i$. It follows that $H \lhd N_i$, that is, $N_i \subseteq N_G(H)$ for all $i = 1, \ldots, r$, and therefore $G \subseteq N_G(H)$. Then H is normal in G. Now, the second equality in (4) means

$$\delta(G/H) = |G/H| - 1$$

and so G/H is also an elementary abelian 2-group by Corollary 1.2 of [3].

This completes the proof. □

Since $d + \frac{n}{d} - 2 \leq n - 1$ for all divisors d of n, we infer the following corollary.

Corollary 2.1. *Let G be a finite solvable group and H be a subgroup of G. Then*

$$d_{L(G)^*}(H) \leq |G| - 1.$$

Moreover, we have

$$|E(L(G)^*)| \leq \frac{1}{2}\, |V(L(G)^*)|(|G| - 1),$$

where $V(L(G)^*)$ and $E(L(G)^*)$ denote the sets of vertices and of edges of the graph $L(G)^*$, respectively.

The second lemma establishes an elementary inequality which will be used in the proof of Theorem 1.1.

Lemma 2.2. *Let p_1, p_2, p_3 be distinct primes and $n_1, n_2, n_3 \in \mathbb{N}^*$. Then*

$$\sum_{i=1}^{3} \frac{p_i^{n_i+1} - p_i}{p_i - 1} \leq \frac{1}{2} \prod_{i=1}^{3} p_i^{n_i}.$$

Proof. Assume $p_1 < p_2 < p_3$ and let $x = p_3^{n_3}$. Then the inequality is equivalent to

$$\left(\frac{1}{2}\, p_1^{n_1} p_2^{n_2} - \frac{p_3}{p_3 - 1}\right) x \geq \frac{p_1^{n_1+1} - p_1}{p_1 - 1} + \frac{p_2^{n_2+1} - p_2}{p_2 - 1} - \frac{p_3}{p_3 - 1}.$$

Since

$$1 < \frac{p_3}{p_3 - 1} \leq \frac{5}{4},$$

it suffices to prove that

$$\left(\frac{1}{2}\, p_1^{n_1} p_2^{n_2} - \frac{5}{4}\right) x \geq \frac{p_1^{n_1+1} - p_1}{p_1 - 1} + \frac{p_2^{n_2+1} - p_2}{p_2 - 1} - 1$$

and since $x \geq p_3$, it suffices to prove that

$$\left(\frac{1}{2}\, p_1^{n_1} p_2^{n_2} - \frac{5}{4}\right) p_3 \geq \frac{p_1^{n_1+1} - p_1}{p_1 - 1} + \frac{p_2^{n_2+1} - p_2}{p_2 - 1} - 1.$$

Let $y = p_2^{n_2}$. Then the above inequality is equivalent to

$$\left(\frac{1}{2}\,p_1^{n_1}p_3 - \frac{p_2}{p_2-1}\right) y \geq \frac{p_1^{n_1+1}-p_1}{p_1-1} + \frac{5}{4}\,p_3 - \frac{p_2}{p_2-1} - 1.$$

Similarly, since

$$1 < \frac{p_2}{p_2-1} \leq \frac{3}{2}$$

and $y \geq p_2$, it suffices to show that

$$\left(\frac{1}{2}\,p_1^{n_1}p_3 - \frac{3}{2}\right) p_2 \geq \frac{p_1^{n_1+1}-p_1}{p_1-1} + \frac{5}{4}\,p_3 - 2.$$

Let $z = p_1^{n_1}$. Then the above inequality is equivalent to

$$\left(\frac{1}{2}\,p_2p_3 - \frac{p_1}{p_1-1}\right) z \geq \frac{3}{2}\,p_2 + \frac{5}{4}\,p_3 - \frac{p_1}{p_1-1} - 2.$$

Similarly, since

$$1 < \frac{p_1}{p_1-1} \leq 2$$

and $z \geq p_1 \geq 2$, it suffices to show that

$$\left(\frac{1}{2}\,p_2p_3 - 2\right) \cdot 2 \geq \frac{3}{2}\,p_2 + \frac{5}{4}\,p_3 - 3,$$

or equivalently

$$p_2p_3 \geq \frac{3}{2}\,p_2 + \frac{5}{4}\,p_3 + 1.$$

This becomes

$$p_2\left(p_3 - \frac{3}{2}\right) \geq \frac{5}{4}\,p_3 + 1.$$

Finally, since $p_2 \geq 3$, it suffices to show that

$$3 \cdot \left(p_3 - \frac{3}{2}\right) \geq \frac{5}{4}\,p_3 + 1,$$

that is, $p_3 \geq \frac{22}{7}$, which is true, completing the proof. $\square$

We are now able to prove our main result.

Proof of Theorem 1.1. Recall that G is a finite solvable group with $|G| = n$ and there exists a subgroup H of G with $|H| = d$, such that $\frac{n}{2} - 1 < d_{L(G)^*}(H)$. Then, by Lemma 2.1, we have

$$\frac{n}{2} - 1 < d + \frac{n}{d} - 2,$$

that is,

$$2d^2 - (n+2)d + 2n > 0. \tag{5}$$

Then either the discriminant $n^2-12n+4$ of (5) is negative, i.e., $n \leq 11$, or $d \in [1, d_1) \cup (d_2, \infty)$, where

$$d_{1,2}=\frac{n+2\mp\sqrt{n^2-12n+4}}{4}.$$

In the first case, it is easy to see that all groups of order at most 11 except C_n for $n=5,\ldots,11$ are solutions of our problem, that is, each of these groups G admits a subgroup H which satisfies $\frac{n}{2}-1<d_{L(G)^*}(H)$. So, in what follows, we will assume that $n \geq 12$. Then, by analytical methods, we get $d_1 \leq 3$ and $d_2 \geq \frac{n}{3}$, implying that $d \in \{1, 2, \frac{n}{2}, n\}$. $\square$

Case 1. $d=1$.

Let r be the number of atoms of the lattice interval between H and G. So, in this particular case, r is the number of atoms in the subgroup lattice of G. Then $r=d_{L(G)^*}(H)>\frac{n}{2}-1$ and we get the solvable groups in Theorem A.

Case 2. $d=2$.

Then $d_{L(G)^*}(H) \geq \frac{n}{2}$. On the other hand, by Lemma 2.1, we have $d_{L(G)^*}(H) \leq \frac{n}{2}$. Thus, in this case, $d_{L(G)^*}(H)=\frac{n}{2}$ and H is normal in G and G/H is an elementary abelian 2-group, say $G/H \cong C_2^s$. Moreover, G is a 2-group and $G' \subseteq H$.

If $G'=1$, we get $G \cong C_2^{s+1}$ or $G \cong C_2^{s-1} \times C_4$, where $s \geq 3$, while if $G'=H$, we get $G'=\Phi(G) \subseteq Z(G)$, i.e, G is a generalized extraspecial 2-group.

Case 3. $d=\frac{n}{2}$.

Let $k=|\mathrm{Max}(H)|$. Then the inequality

$$1+k=d_{L(G)^*}(H)>\frac{n}{2}-1$$

means $k \geq \frac{n}{2}-1$ and so $H \cong C_2^t$ for some positive integer t by Theorem B (b). It follows that $n=2^{t+1}$ and $i_2(G) \geq i_2(H)=2^t$, i.e., $i_2(G)>\frac{|G|}{2}-1$, where $i_2(G)$ denotes the number of involutions of G. Thus we get the groups classified by Wall [14], i.e., the groups of types (I)-(IV) in Theorem A.

Case 4. $d=n$.

Then

$$k=d_{L(G)^*}(H)>\frac{n}{2}-1. \tag{6}$$

Let $n=p_1^{n_1}\cdots p_u^{n_u}$ be the decomposition of n as a product of distinct primes and assume that $p_1^{n_1}=\min\{p_i^{n_i}: 1 \leq i \leq u\}$. Using Theorem B (e), we obtain

$$k \leq \frac{p_1^{n_1}-1}{p_1-1}+\sum_{i=2}^{u}\frac{p_i^{n_i+1}-p_i}{p_i-1} \leq \sum_{i=1}^{u}\frac{p_i^{n_i+1}-p_i}{p_i-1}-1.$$

Now we prove by induction that for all $u \geq 3$, we have

$$\sum_{i=1}^{u}\frac{p_i^{n_i+1}-p_i}{p_i-1} \leq \frac{n}{2}$$

and so $k \leq \frac{n}{2} - 1$, contradicting (6). For $u = 3$, the inequality holds from Lemma 2.3. Assume that it holds for an arbitrary $u \geq 3$. By the inductive hypothesis, we get

$$\sum_{i=1}^{u+1} \frac{p_i^{n_i+1} - p_i}{p_i - 1} \leq \frac{1}{2}\, p_1^{n_1} \cdots p_u^{n_u} + \frac{p_{u+1}^{n_{u+1}+1} - p_{u+1}}{p_{u+1} - 1}. \tag{7}$$

The fact that the right side of (7) is at most $\frac{1}{2}\, p_1^{n_1} \cdots p_u^{n_u} p_{u+1}^{n_{u+1}}$ is equivalent with

$$\frac{p_{u+1}^{n_{u+1}+1} - p_{u+1}}{p_{u+1} - 1} \leq \frac{1}{2}\, p_1^{n_1} \cdots p_u^{n_u} \left(p_{u+1}^{n_{u+1}} - 1\right),$$

i.e., with

$$\frac{p_{u+1}}{p_{u+1} - 1} \leq \frac{1}{2}\, p_1^{n_1} \cdots p_u^{n_u},$$

which is obviously true.

Thus $u \leq 2$. Also, we have $2 \mid n$ and $\Phi(G) = 1$ by Theorem B (c), and the condition $k \geq n/2$.

If $u = 1$, then G is obviously an elementary abelian 2-group. Assume that $u = 2$. Then $n = p_1^{n_1} p_2^{n_2}$ with $p_1 = 2$ or $p_2 = 2$. For $p_1 = 2$, the inequality $n/2 \leq k$ becomes

$$2^{n_1 - 1} p_2^{n_2} \leq 2^{n_1} - 1 + \frac{p_2^{n_2+1} - p_2}{p_2 - 1}.$$

Then $n_1 = 1$, i.e., $n = 2 \cdot p_2^{n_2}$. Let P_2 be a Sylow p_2-subgroup of G. Since P_2 is normal in G, we obtain $\Phi(P_2) = 1$, that is, $P_2 \cong C_{p_2}^{n_2}$. This shows that $G \cong C_{p_2}^{n_2} \rtimes C_2$. For $p_2 = 2$, the inequality $n/2 \leq k$ becomes

$$p_1^{n_1} 2^{n_2 - 1} \leq \frac{p_1^{n_1} - 1}{p_1 - 1} + 2^{n_2+1} - 2,$$

implying that $p_1 = 3$ and $n_1 = 1$, i.e., $n = 3 \cdot 2^{n_2}$ with $n_2 \geq 2$. Clearly, G is not A_4, and since $\Phi(G) = 1$, the group G possesses a normal subgroup of index 2. It follows that $\mathbf{O}^2(G) < G$ and so

$$|\mathrm{Max}_2(G)| \leq 2^{n_2} - 1$$

by Theorem B (d). Since G is solvable, its maximal subgroups are of prime power index, which leads to

$$k = |\mathrm{Max}_3(G)| + |\mathrm{Max}_2(G)|.$$

If G has a unique Sylow 2-subgroup, i.e., $|\mathrm{Max}_3(G)| = 1$, then we get

$$3 \cdot 2^{n_2 - 1} \leq k = 1 + |\mathrm{Max}_2(G)| \leq 2^{n_2},$$

a contradiction. If G has three Sylow 2-subgroups, i.e., $|\mathrm{Max}_3(G)| = 3$, then we get

$$3 \cdot 2^{n_2 - 1} \leq k = 3 + |\mathrm{Max}_2(G)| \leq 2^{n_2} + 2,$$

that is, $n_2 = 2$. Thus $n = 12$ and the unique solution of our problem is $G \cong D_{12}$.

The proof of Theorem 1.1 is now complete. □

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